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H. J. KUSHNER
Y. BAR-NESS

ANALYSIS OF A PHASE-LOCKED LOOP AND A GENERAL FEEDBACK SYSTEM WITH A LIMITER

OCTOBER 1979

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Lefschetz Center for Dynamical Systems

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A phase-locked loop (PLL) with a hard limiter and wide-band noise input is analyzed. A powerful method introduced in Kushner's "Diffusion approximations to output processes of nonlinear systems with wide band inputs, and applications" for obtaining a close diffusion approximation to a system with the limiter greatly improves the tracking and acquisition properties over the classical PLL. This is demonstrated theoretically and supported by simulation results. Similar results hold for a standard feedback control problem with a limiter at the beginning of the forward loop. The results are a clear

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demonstration of the value of the new techniques of analysis which are employed. The techniques are applicable to other communications systems with nonlinearities and wide-band inputs.

ANALYSIS OF A PHASE-LOCKED LOOP AND A GENERAL FEEDBACK SYSTEM WITH A LIMITER

H. J. Kushner
Y. Bar-Ness

1. Introduction

Consider a (linear or nonlinear) dynamical system with wide-band noise input. In communication theory, it is often of considerable interest to approximate such systems by diffusion models so that, e.g., Fokker-Planck and other "diffusion" process techniques can be used. In [1], a general and powerful method for doing this was developed. Roughly, the input noise process is parametrized by ε and as $\varepsilon \to 0$, the bandwidth (BW) goes to infinity, while the power per unit BW approaches a constant. The process which corresponds to the limit of the sequence of measures of the output or system state processes $\{x^{\varepsilon}(\cdot)\}$ is found by techniques of weak convergence theory.

The method is particularly useful when the noise (and/or signal) is processed nonlinearly. In order to illustrate the method, three applications were developed in [1]: (a) a phase locked loop (PLL); (b) an adaptive antenna array; and (c) output of a hard limiter followed by a filter. Difficulties with some of the heuristic methods were avoided. In fact, many of the more heuristic arguments for problems (a), (b) use (implicitly or explicitly) a "wide band assumption".

In this paper a very interesting and more difficult new application, essentially a combination of (a), (c) above, is discussed. There is a well-developed classical theory of PLLs [2], [3]. Recently, a strong practical interest in PLLs that use either nonlinear filters or static nonlinearities has developed [4], [5] and there have also been prior indications (based on simulation studies) that use of certain nonlinearities could actually improve the loop's performance. The improvements in performance that are possible with the use of a hard limiter type nonlinearity will be demonstrated theoretically, via the method of [1], and confirmed by simulation. It seems to the authors that there are no current alternative methods for the treatment of such problems in a rigorous or readily rigorizable but intuitively convincing manner. A number of interesting ideas and results appear in [5], but the method requires an "equivalent white noise" input, and would seem to have great difficulty (even with a heuristic treatment) with the problem treated here. However, the general conclusions of [5] are consistent with our own.

Our development will be rigorous in part and heuristic in part. It is partly heuristic because we seek to simplify the analysis to avoid some of the more onerous calculations. But even there, the development will be guided by the intuition developed in [1] for the properties of the simpler systems treated there. The point is that, used even heuristically, the method provides an interesting and powerful tool for getting the appropriate diffusion approximations to many nonlinear systems whose (even formal) treatment would be much more difficult by alternative methods.

ANALYSIS OF A PHASE-LOCKED LOOP

AND A GENERAL FEEDBACK SYSTEM WITH A LIMITER

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Y. Bar-Ness** (Senior Member, IEEE)

Abstract

A phase-locked loop (PLL) with a hard limiter and wide-band noise input is analyzed. A powerful method introduced in [1] for obtaining a close diffusion approximation to a system with a wide-band input is applied. For small input noise power per unit bandwidth, the limiter greatly improves the tracking and acquisition properties over the classical PLL. This is demonstrated theoretically and supported by simulation results. Similar results hold for a standard feedback control problem with a limiter at the ɜginning of the forward loop. The results are a clear demonstration f the value of the new techniques of analysis which are employed. The techniques are applicable to other communications systems with nonlinearities and wide-band inputs.

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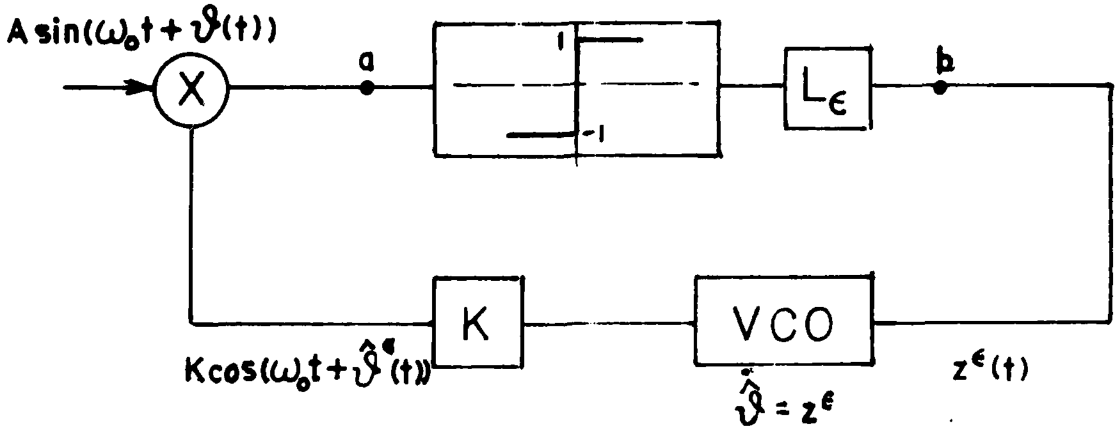
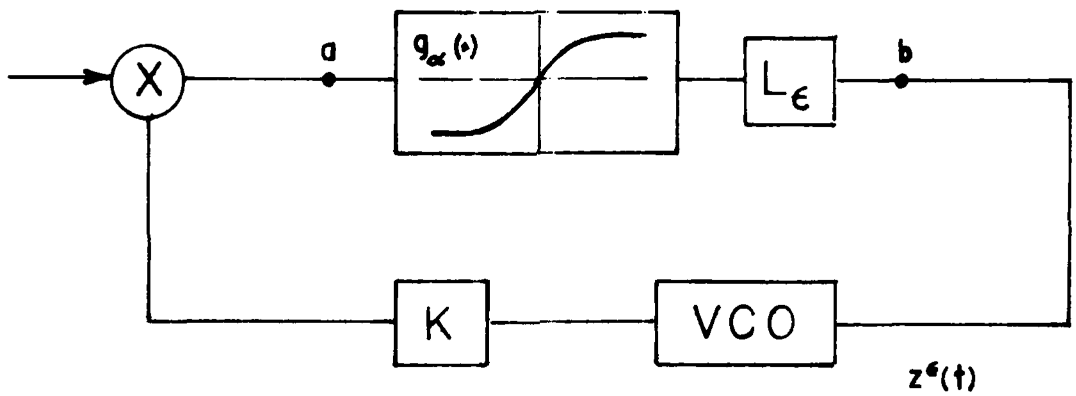
The results and method are also quite important in control theory. There the filter and VCO would or could be replaced by general linear or nonlinear dynamical systems and the multiplier by an adder. In fact, for this case, our results are rigorous, and indicate the important role which a hard limiter can play in improving a systems noise immunity when the noise is small but the bandwidth is large.

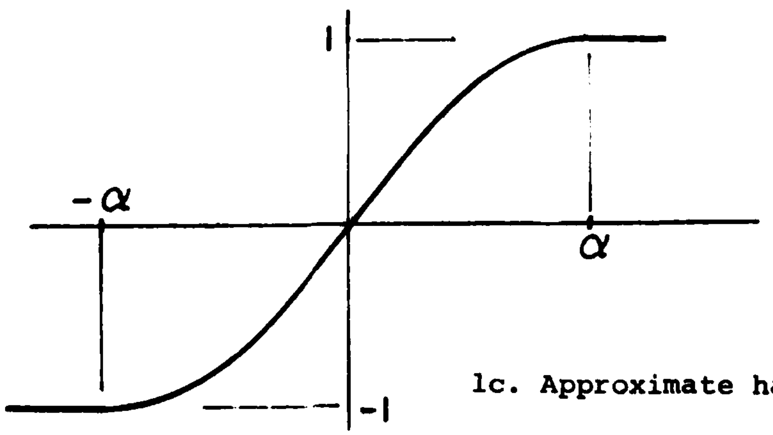
We are concerned with the systems of Figures 1 and 2, the first (resp., second) being a first- (arbitrary, resp.) order PLL. By suitable choice of D, E, C, G in Figure 2, any zonal filter can be well approximated. In Figures 1a, 2a we have a hard limiter, and in Figures 1b, 2b an approximation to a hard limiter. $n^{\varepsilon}(\cdot)$ denotes the input noise, $\hat{\theta}^{\varepsilon}(\cdot)$ the estimate of the "signal" phase $\theta(\cdot)$ of the input signal $A \sin(\omega_0 t+\theta(t))$. The general type of approximation $g_{\alpha}(\cdot)$ which is used for $g(\cdot)$, the hard limiter, is graphed in Figure 1c. The treatment of the general case of Figure 2 is almost identical to that of the simpler case of Figure 1; one only carries a few extra terms (which are not hard to handle in the development). In the interest of simplicity, derivations for the first-order case will be given; the results for the general case will be only stated. The results indicate some rather surprising and important advantages of the use of the limiter; for small noise power, the acquisition range and tracking ability are increased, and acquisition time decreased, as compared with the case with no limiter. These



1a. PLL with hard limiter

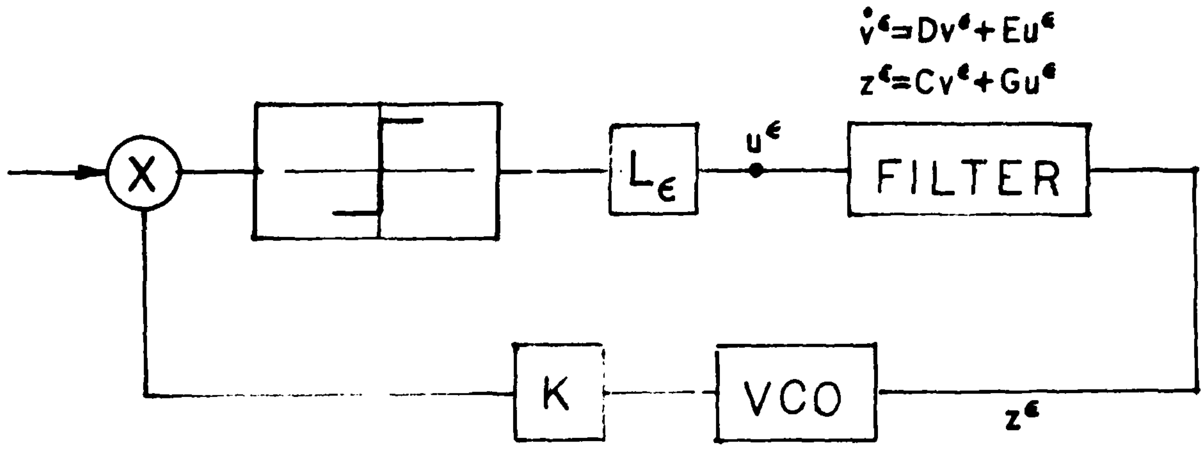


1b. PLL with approximate hard limiter

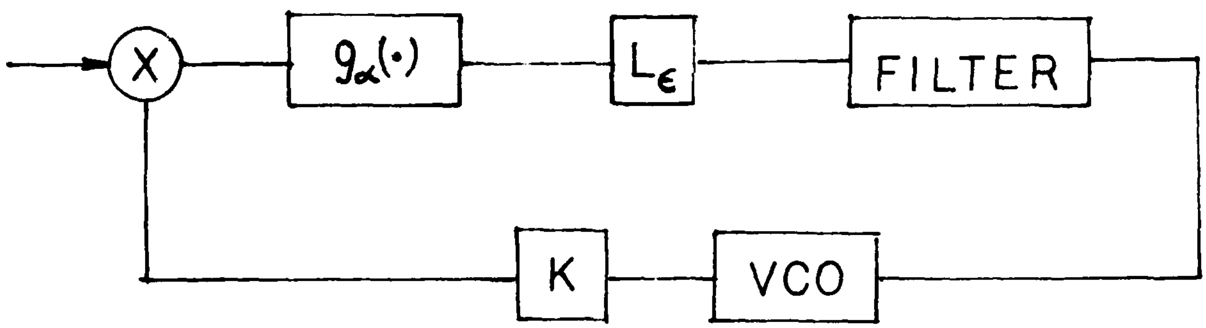


1c. Approximate hard limiter

FIG. 1, FIRST ORDER PLL



2a. General PLL with hard limiter



2b. General PLL with approximate hard limiter.
Same inputs as in Fig. 1

FIG. 2, GENERAL ORDER PLL

advantages are borne out by simulations, and have important analogues in the automatic control problem also.

In Section 2 various assumptions are given and in Section 3 a result of [1] is applied directly to get a limit approximation in a heuristic manner. The limit seems to capture the essential properties, and it is compared with the comparable equation for the classical first-order PLL. Section 4 discusses some simulation results. The Appendix contains various parts of the detailed development.

Note that it makes no mathematical or physical sense for $n^{\varepsilon}(\cdot)$ to be a white Gaussian noise. Owing to the properties of white Gaussian noise, the output of the limiter would be zero. So we must deal with a wide band input.

2. Assumptions

The noise model. For the sake of concreteness in the calculation and in order to be able to use the results of [1], Section 6, it is supposed that $n^{\varepsilon}(t) = y^{\varepsilon}(t)/\varepsilon$ where $y^{\varepsilon}(t) = y(t/\varepsilon^2)$ and $y(\cdot)$ is a stationary Gaussian process with correlation function $\sigma^2 \exp - a|\tau|$, $a > 0$. Reasons for this scaling are discussed in [1], Section 2 and in [6]. It is a convenient and common way of getting a noise process $n^{\varepsilon}(\cdot)$ whose spectrum converges (as $\varepsilon \to 0$) to that of a white noise with a constant power per unit bandwidth, namely $2\sigma^2/a$. It is, perhaps, the simplest model that could be used for our wide band approximation technique. In order for $n^{\varepsilon}(\cdot) \to$ "white Gaussian noise" or $\int_0^t n^{\varepsilon}(s)ds \to$ Wiener process as $\varepsilon \to 0$, a contraction of the time scale and an increase in the amplitude of $y(\cdot)$ is required (as $\varepsilon \to 0$). This is discussed further in the reference. The particular form of $y(\cdot)$ was chosen partly for convenience, in that it enables us to use an available result. Any Gaussian process whose correlation function decreases exponentially can be used, but the $\sqrt{2 \ln 2/a}$ coefficient in (3.1) and in derivative expressions would be replaced by a different coefficient although the general result of the sequel (the improved tracking ability for small power per unit BW with the use of the limiter) would remain true.

In the classical case (e.g., [2, 3]), a somewhat similar "limit" assumption is made; although the noise $n^{\varepsilon}(\cdot)$ is not explicitly parametrized, and no limits are explicitly taken, in the various approximations which are made it is implicitly or explicitly assumed that the BW is "very wide" (this assumption is used in an intuitive or formal way only, in the classical case). For the system without the limiter, much weaker assumptions on $n^{\varepsilon}(\cdot)$ can be used [1] without over-complicating the development.

<u>The scaling</u> L_{ε}. If $L_{\varepsilon} = L$, a number not depending on ε, then a heuristic "averaging" type of analysis implies that the "<u>effective gain</u>" of either the $L_{\varepsilon}g(\cdot)$ or $L_{\varepsilon}g_{\alpha}(\cdot)$ nonlinearities goes to zero as $\varepsilon \to 0$ (or, equivalently, as BW $\to \infty$). Thus in any "limit" analysis, L_{ε} must increase as ε decreases. In any particular practical system, operating in a fixed signal and noise environment, one particular value of $L = L_{\varepsilon}$ will of course be used. Other things remaining equal, as the BW of the input process increases (without the power per unit BW degenerating to zero, in

the "pass band"), this value of L will have to increase. How this value is to be chosen in any particular application is not important here. What is important is that ϵL_ϵ must converge to a constant $L > 0$ as $\epsilon \to 0$. (See Appendix.) This property holds if $y(\cdot)$ is any stationary Gaussian process. So $L_\epsilon = L/\epsilon$ will be used in the limit analysis. If $\epsilon L_\epsilon \to 0$ as $\epsilon \to 0$, the limit system will be an open circuit.

3. A Heuristic Analysis

Consider the system of Figure 1a. In [1], Section 6, the limit equation (as $\epsilon \to 0$) for the output process $x^\epsilon(\cdot)$ of a hard limiter of level L with input $s(t)+n^\epsilon(t)$ ($s(\cdot)$ being piecewise continuous, but arbitrary otherwise - it is an input signal) is derived. The limit process has the same distribution as the diffusion

$$(3.1)\quad dx = L\sqrt{2/\pi}\,\frac{s(t)}{\sigma}\,dt + L\sqrt{2\ln 2/a}\,dB,$$

where $B(\cdot)$ is (and always will be in this paper) a standard Wiener process. The following heuristic derivation, for which greater justification is provided in the Appendix, captures the essential effects.

At point (a) of Figure 1a, the signal is

$$(3.2)\quad KA\sin(\omega_0 t+\theta(t))\cos(\omega_0 t+\hat{\theta}^\epsilon(t)) + K\cos(\omega_0 t+\hat{\theta}^\epsilon(t))n^\epsilon(t)$$

$$\equiv a^\epsilon(t).$$

Interpret $v/|v| = 0$ if $v = 0$ (let sign $0 = 0$ for definiteness).

At (b) in Figure 1a, we have

$$(*) \qquad z^{\varepsilon}(t) = \dot{\hat{\theta}}^{\varepsilon}(t) = \frac{L}{\varepsilon}\,\mathrm{sign}[a^{\varepsilon}(t)/|\cos(\omega_0 t+\hat{\theta}^{\varepsilon}(t))|].$$

Supposing that the $\hat{\theta}^{\varepsilon}(\cdot)$ variations are "slow" compared with those of $n^{\varepsilon}(\cdot)$, drop the $\hat{\theta}^{\varepsilon}(\cdot)$ terms in the noise coefficient $n^{\varepsilon}(t)\cos(\omega_0 t+\hat{\theta}^{\varepsilon}(t))/|\cos(\omega_0 t+\hat{\theta}^{\varepsilon}(t))|$ of the argument of the function on the right side of (*). This is the main simplifying assumption and will be discussed further below. There is no need for this step if the multiplier in Figures 1 or 2 is replaced by an adder (whatever replaces the VCO) as in the control problem. The noise term in the argument of the function on the right side of (*) is now $\mathrm{sign}(\cos\omega_0 t)n^{\varepsilon}(t)$. On substituting $t/\varepsilon^2 \to t$, this becomes $\mathrm{sign}(\cos\omega_0\varepsilon^2 t)\cdot y(t)/\varepsilon$. For small ε and over a finite interval, the statistics of this process are essentially those of $y(\cdot)/\varepsilon$. As will be seen in the Appendix, it is the statistics in the new scale which determine the correct limit. Thus, since we are concerned with small ε, this suggests that the $\mathrm{sign}(\cos\omega_0\varepsilon^2 t)$ term can be replaced by unity without altering the limit. Finally, making these substitutions we have the formal approximation:

$$\mathrm{sign}(a^{\varepsilon}(t)) \approx \mathrm{sign}[A\sin(\omega_0 t+\theta(t))\mathrm{sign}(\cos(\omega_0 t+\hat{\theta}^{\varepsilon}(t)) + y^{\varepsilon}(t)/\varepsilon]$$

$$\equiv \mathrm{sign}[s^{\varepsilon}(t) + y^{\varepsilon}(t)/\varepsilon],$$

where $s^{\varepsilon}(\cdot)$ is defined in the obvious way. With this approximation at point (b) in Figure 1a, we have

$$(**)\qquad z^{\varepsilon}(t) = \dot{\hat{\theta}}^{\varepsilon}(t) = \frac{L}{\varepsilon}\,\mathrm{sign}[s^{\varepsilon}(t) + y^{\varepsilon}(t)/\varepsilon].$$

Next, ignoring the fact that $s^{\varepsilon}(\cdot)$ depends on $\hat{\theta}^{\varepsilon}(\cdot)$, and formally using the limit result (3.1), yields that the limit (as $\varepsilon \to 0$) of $\{\hat{\theta}^{\varepsilon}(\cdot)\}$ has the same distributions as the process $\hat{\theta}(\cdot)$ given by (we replace the $\hat{\theta}^{\varepsilon}(\cdot)$ inside the sign by the "limit" $\hat{\theta}(\cdot)$ also)

$$(3.3)\qquad d\hat{\theta}(t) = L\sqrt{2/\pi}\,\frac{A\sin(\omega_0 t+\theta(t))\ \mathrm{sign}[\cos(\omega_0 t+\hat{\theta}(t))]}{\sigma}\,dt + L\sqrt{\frac{2\ln 2}{a}}\,dB.$$

Greater justification for the limit representation (3.3) will be given in the Appendix. Equation (3.3) has a unique solution (in the sense of measure), despite the discontinuity [8]. Also, the argument concerning setting the sign $\cos(\omega_0 t)$ term equal to unity can be justified in the sense that it does not affect the limits. The limit (3.3) can be justified if we start with (**), use the approximation $g_\alpha(\cdot)$ instead of† $g(\cdot)$, and then let $\alpha \to 0$ as $\varepsilon \to 0$. See the Appendix. The main difficulty lies in ignoring the $\hat{\theta}^{\varepsilon}$ dependence of the coefficients of $n^{\varepsilon}(t)$. This dependence gives rise to what is usually called the "correction term", an additional drift term which should appear in (3.3). Purely formal calculations suggest that this term "oscillates with ω_0", and that it would average out if either ω_0 is large or a narrow band filter is used, as in Figure 2. This is, of course, what happens to the correction term

†The reason for the use of the approximate hard limiter $g_\alpha(\cdot)$ in the analysis in the Appendix is that our approach requires a differentiable nonlinearity. In the Appendix, we get the limit when $\varepsilon \to 0$ and $\alpha \to 0$ as $\varepsilon \to 0$. This is a reasonable point of view.

in the standard case [1, Section 4], but here we are not yet able to explicitly evaluate this term.

3.1 Approximation of (3.3). Let us now approximate (3.3) by a simpler system. The justification of the approximation (for large ω_0) appears in the Appendix. First expand $q(a) \equiv \mathrm{sign}(\cos u) = \sum_{k=1}^{\infty} q_k \cos ku$ over $[-\pi,\pi]$, where $q_k = \frac{2}{\pi}\int_0^{\pi} f(u) \cos ku\, du =$ $(4/\pi k)\sin k\pi/2$. Thus $q_{2k} = 0$, all k, and

$$\mathrm{sign}(\cos(\omega_0 t+\hat{\theta})) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \cos(2k+1)(\omega_0 t+\hat{\theta})$$

and

$$\begin{aligned}\sin(\omega_0 t+\theta)q(\omega_0 t+\hat{\theta}) &= \left(\frac{4}{\pi}\right)\frac{1}{2}[\sin(\theta-\hat{\theta})+\sin(2\omega_0 t+\theta+\hat{\theta})] \\ &+ \frac{4}{\pi}\sin(\omega_0 t+\theta)\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)}\cos(2k+1)(\omega_0 t+\hat{\theta}) \\ &\equiv \frac{2}{\pi}(\sin(\theta-\hat{\theta})) + F(\omega_0,t,\theta,\hat{\theta}).\end{aligned}$$

Now, we can take either of two approaches. The simplest is simply to assert that F, which contains the "high-frequency" terms, will be filtered out. An alternative, which is developed in the Appendix, is to use the averaging method of [1] (see Section 4 there) again, but where the $\hat{\theta}(\cdot)$ of (3.3) is now parametrized by ω_0 and to show that, as $\omega_0 \to \infty$, the $\hat{\theta}(\cdot)$ converge weakly (in $C[0,\infty)$; see Appendix) to the solution of the "averaged equation"

$$(3.4)\quad d\hat{\theta} = L\left(\frac{2}{\pi}\right)^{3/2} \frac{A\sin(\theta-\hat{\theta})}{\sigma}\,dt + L\sqrt{\frac{2\ln 2}{a}}\,dB.$$

Equation (3.4) is the desired limit equation. Suppose, for purposes of representation, that $\theta(\cdot)$ is differentiable. Then, with $\phi = \theta-\hat{\theta}$,

$$(3.5)\quad d\phi = \dot{\theta}dt - L\left(\frac{2}{\pi}\right)^{3/2} \frac{A\sin\phi}{\sigma}\,dt - L\sqrt{\frac{2\ln 2}{a}}\,dB.$$

See Appendix 3 for the details.

<u>The general filter case</u>. The case of Figure 2 can be dealt with similarly, to obtain

$$(3.6)\quad d\phi = \dot{\theta}dt - Cvdt - LG\left[\frac{A}{\sigma}\left(\frac{2}{\pi}\right)^{3/2}\sin\phi\,dt + \sqrt{\frac{2\ln 2}{a}}\,dB\right],$$

$$dv = Dvdt + LE\left[\frac{A}{\sigma}\left(\frac{2}{\pi}\right)^{3/2}\sin\phi dt + \sqrt{\frac{2\ln 2}{a}}\,dB\right].$$

<u>3.2 Limit equations for the classical case (no limiter)</u>. For comparison purposes, only the first-order case will be studied. The classical derivation appears in [2], [3]. A derivation of the limit as input BW $\to\infty$ (an assumption also made for the classical derivation) is in [1], Section 4. In that reference, $n^{\varepsilon}(\cdot)$ was a more general process than used here. Specializing to the $n^{\varepsilon}(\cdot)$ used here, we have [1, eqn. (4.3)]: For fixed ω_0, $\phi^{\varepsilon}(\cdot) = \theta(\cdot)-\hat{\theta}^{\varepsilon}(\cdot)$ converges weakly (in $C[0,\infty)$) to the solution of the diffusion equation

(3.7) $$d\phi(t) = \dot{\theta}(t)dt - \frac{AK}{2}[\sin \phi(t) + \sin(\theta(t)+\hat{\theta}(t)+2\omega_0 t)]dt$$

$$+ \frac{\sigma^2 K^2}{a} \cos(\omega_0 t+\hat{\theta}(t)) \sin(\omega_0 t+\theta(t))dt$$

$$- K\sqrt{\frac{2}{a}}\,\sigma \cos(\omega_0 t+\hat{\theta}(t))dB \;.$$

As $\omega_0 \to \infty$, $\phi(\cdot)$ (parametrized by ω_0) converges weakly (in $C[0,\infty)$; see [1]) to the solution of

(3.8) $$d\phi = \dot{\theta}dt - \frac{AK}{2} \sin \phi \; dt - K \frac{\sigma}{\sqrt{a}} \; dB.$$

The third term on the right of (3.7), the so-called "correction" term, arises due to the non-independence of $n^\varepsilon(t)$ and $\hat{\theta}^\varepsilon(t)$. The classical derivation ignores this dependence. But, as $\omega_0 \to \infty$, the term is "averaged out", whether or not a filter is used in the PLL. In fact, by our neglect of the $\hat{\theta}^\varepsilon$ dependence of the coefficient of $n^\varepsilon(\cdot)$ in the argument leading to (3.3), the correction term does not appear.

3.3 Comparison of (3.5) and (3.8). L in (3.5) plays the role of K in (3.8). The salient and surprising difference concerns the role of σ. In (3.8), it affects the noise term (proportionally). In (3.5), it affects the drift term (inversely). It is thus expected that for small σ, the limiter would improve the acquisition and tracking properties of the PLL.

Let $\dot{\theta} = \omega_1$, a constant, and linearize the sin in both (3.5) and (3.8) about $\phi \equiv 0$. Then the asymptotic (linearized) mean square values $E(\theta-\hat{\theta})^2$ are

$$(3.9)\quad \text{(limiter (3.5))}\ \omega_1^2 \left(\frac{\pi}{2}\right)^3 \frac{\sigma^2}{L^2A^2} + L\ (\ln 2)\left(\frac{\pi}{2}\right)^{3/2} \frac{\sigma}{aA}$$

$$= 3.87\ \frac{\omega_1^2\sigma^2}{L^2A^2} + 1.36\ \frac{L\sigma}{Aa}\ ;$$

$$(3.10)\quad \text{(no limiter (3.8))}\ \frac{4\omega_1^2}{K^2A^2} + \frac{K\sigma^2}{Aa}\ .$$

The major shortcoming of the first-order system with no limiter is its poor acquisition ability, irrespective of σ, a shortcoming which the use of the limiter overcomes when σ is small, and the bandwidth large. The same relative advantage holds for the general case of Figure 2, as we can easily see from (3.6).

3.4 An alternative heuristic development. In order to gain some intuitive feeling for the effects of different approximations of the input to the limiter, let us now suppose that the input (3.2) to the limiter is replaced by the "natural approximation" (not always so natural in the nonlinear case)

$$(3.11)\quad \frac{AK}{2} \sin\ (\theta(t)-\hat{\theta}^{\varepsilon}(t)) + \frac{K}{\sqrt{2}}\, n^{\varepsilon}(t),$$

where we get the $\sqrt{2}$ by "averaging" the square of the coefficient of $n^{\varepsilon}(\cdot)$ and then taking its square root, while ignoring the coefficient's $\hat{\theta}^{\varepsilon}$ dependence. Then, by using the argument which led to (3.3), we get the limit equation

$$(3.12)\qquad d\phi = \dot{\theta}dt - AL\frac{\sin\phi}{\sqrt{\pi}\,\sigma}dt - L\sqrt{\frac{2\ln 2}{a}}\,dB.$$

The ratio of the drift coefficient in (3.5) to that in (3.12) is $2\sqrt{2}/\pi \approx 0.9$. Thus, the process (3.12) would be expected to have smaller errors - but not by much - and, otherwise, the same qualitative properties as (3.5) has.

3.5 An automatic control problem. The results are directly applicable to the typical feedback situation of Figure 3. $s(\cdot)$ is the signal or informational component in the input, $n^{\varepsilon}(\cdot)$ is, again, the wide-band noise of Section 2 and $F(\cdot)$ and $G(\cdot)$ are smooth functions with at most a linear growth rate. The same formal argument which led to (3.3) yields (no dropping of the $\hat{\theta}^{\varepsilon}(\cdot)$ in the noise coefficient is required, as was done in the PLL case in order to get (**) there) that $\{v^{\varepsilon}(\cdot)\}$ converges weakly to the solution of

$$(3.13)\qquad dv = F(v)dt + LD\left[\frac{(s(t)+G(v))}{\sigma}\sqrt{\frac{2}{\pi}}\,dt + \sqrt{\frac{2\ln 2}{a}}\,dB\right].$$

The $1/\sigma$ effect in (3.13) would seem to be a rather important result. The result is not entirely expected and undoubtedly has many useful applications to control problems when the noise is small in power per unit BW, but of wide BW. For this problem, the method in the Appendix rigorously justifies (3.13).

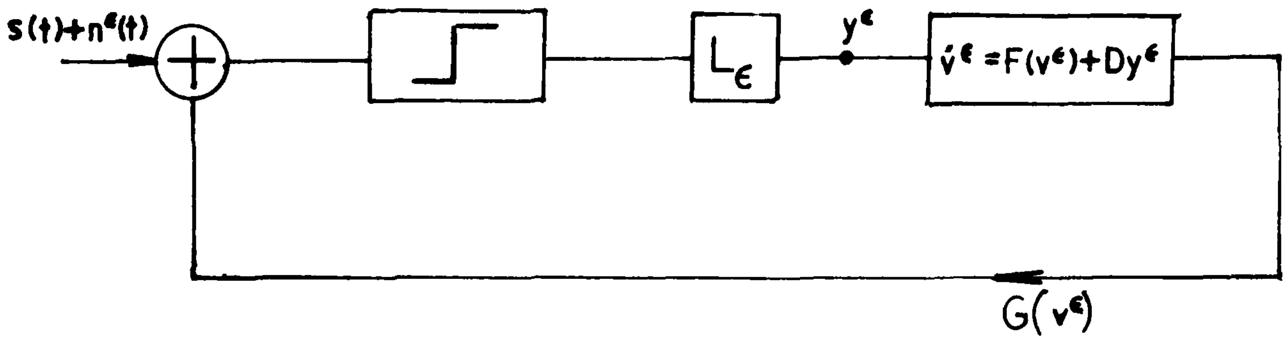


FIG. 3, THE AUTOMATIC CONTROL PROBLEM

4. Simulation Results

The main purposes of the (digital) simulations were to verify the effects of the $1/\sigma$ in the drift term of the limit equation for the "hard limiter" case, and to get some rough idea of the "rate" of convergence to the limit, as well as a rough comparison with the case of no limiter. Consistent with these goals, and in order to avoid (time-scale and length of simulation) problems in simulating either (3.3) or the original system with the ω_0 present, we simulated the simpler cases of Section 3.4, where (3.11) appears at the limiter input. Typical samples of the results appear in Tables 1 and 2. The quantities are sample means and variances over a 300 time unit interval.

Discussion. From Table 1, it can be seen that the limit equation results are very close to those of the actual systems. The results are closer for the no-limiter case, and for the larger value of σ. The presence of the limiter nonlinearity seems to slow down the convergence. (Of course, even without the limiter, the system is nonlinear, but "more softly so".) The quality of the tracking ability is demonstrated by Table 2. For $\epsilon \leq 1/2$, the actual system performs quite close to the limit system. It is remarkable that even for larger values of ϵ, the performance with the limiter can be more favorable (than that without the limiter) than suggested by merely comparing the true limits. The relative contributions of the means and variances are (qualitatively) exactly as expected. The decrease in mean square error as σ decreases, as suggested by the $1/\sigma$ factor in (3.5), is clearly demonstrated by Table 2. As noted above, the useful effect of the limiter (as quantified by the $1/\sigma$ factor in the limit equation) is very clear, even for $\varepsilon = 1$. Owing to the fact that the maximum slope of sin u is at $u = 0$, the actual data are all smaller than those predicted by (3.1) (and the appropriate analogue of (3.8) for the system (3.12)).

The linearizations (3.10) and the analogue of (3.9) for the system (3.12) give conservative estimates of the numbers in the tables. This is because the linearization results are obtained by replacing sin ϕ by ϕ in the dynamics, a "stabilizing" change.

	No limiter			Limiter	
	sample mean	sample variance		sample mean	sample variance
			$\sigma = .5$		
limit	.11	.30		.06	1.0
$\varepsilon = 1$	.11	.20		.02	.38
$\varepsilon = 1/2$	-.04	.27		.02	.82
			$\sigma = .25$		
limit	.05	.068		-.07	.40
$\varepsilon = 1$	.05	.046		.05	.12
$\varepsilon = 1/2$	-0.2	.062		-.02	.23

Table 1. First-order loop, $\theta(t) = \theta_0$, a constant.

	No limiter			Limiter		
			$\sigma = 1/2$			
	sample mean	sample variance	sample mean square error	sample mean	sample variance	sample mean square error
limit	.59	.46	.81	.26	.95	1.02
$\varepsilon = 1$	.62	.24	.62	.15	.47	.49
$\varepsilon = 1/2$	.58	.58	.84	.23	.91	.96
			$\sigma = 1/4$			
limit	.55	.060	.37	.14	.34	.36
$\varepsilon = 1$	.54	.045	.33	.086	.10	.10
$\varepsilon = 1/2$	.57	.07	.40	.12	.26	.27
$\varepsilon = 1/4$				.14	.29	.31

Table 2. First-order loop, $\theta(t) = \theta(0) + t/4$.

Appendix 1: Simplifying the Problem

Modifying the problem. The given problem will be modified in several ways, some insignificant, and Theorem 1 of [1] applied to illustrate the type of argument required to justify the limit equations. First, owing to mathematical necessity, we must work with a smooth but arbitrarily close approximation to the hard limiter. I.e., the system of Figure 1 will be used. The object is to show that the diffusion approximation is arbitrarily close if ε is small and g_α is close enough to a hard limiter. When we discuss limits as $\varepsilon \to 0$ and $\alpha \to 0$ (i.e., as BW $\to \infty$ and $g_\alpha(\cdot) \to$ hard limiter), we mean that $\varepsilon \to 0$, $\alpha \to 0$ simultaneously with $\varepsilon \to 0$ "sufficiently faster"; in particular that $\sqrt{\varepsilon}/\alpha \to 0$ as $\varepsilon \to 0$. This particular ratio is due to the method used and is probably not generic in any sense. $g_\alpha(\cdot)$ is required to have a continuous first derivative, which is bounded by C/α in $[-\alpha,\alpha]$ (the derivative is zero out of this interval), where C always denotes a constant, the value of which may change from usage to usage.

Let $y_0(\cdot)$ denote a stationary Gaussian process, independent of $y(\cdot)$, and with covariance $\sigma_0^2 \exp -at$. Define $y_0^\varepsilon(t) = y_0(t/\varepsilon^2)$ and $n_0^\varepsilon(t) = y_0^\varepsilon(t)/\varepsilon$. In order to "smoothe" the sign(ccs) in (3.3), we add $Kn_0^\varepsilon(\cdot)$ to the signal at the input to the nonlinearity. The process $Kn_0^\varepsilon(\cdot)$ converges to a white Gaussian noise with spectral density $K^2\sigma_0^2/a$ as $\varepsilon \to 0$ and (as will be seen) has negligible effect on the result when σ_0^2 is small. So the modification is harmless and is mainly for mathematical convenience.

With this addition, the input to the nonlinearity $g_\alpha(\cdot)$ is

$$\text{(A1)}\qquad a_0^\varepsilon(t) \equiv KA\,\sin(\omega_0 t+\theta(t))\cos(\omega_0 t+\hat\theta^\varepsilon(t))$$
$$+\,K(\cos(\omega_0 t+\hat\theta^\varepsilon(t))n^\varepsilon(t)+Kn_0^\varepsilon(t).$$

Two additional alterations will be made, the first one harmless and merely for convenience in the analysis, the other more important. Define $\beta^\varepsilon(t) = [\sigma_0^2 + \sigma^2\cos^2(\omega_0 t+\hat\theta^\varepsilon(t))]^{-1/2}/K$ and write

$$\text{(A2)}\qquad \dot{\hat\theta}^\varepsilon(t) = \frac{L}{\varepsilon}\, g_\alpha[\beta^\varepsilon(t)a_0^\varepsilon(t)] + \frac{L}{\varepsilon}\,\delta g_\alpha(t),$$

where

$$\delta g_\alpha(t) = g_\alpha(a_0^\varepsilon(t))-g_\alpha(\beta^\varepsilon(t)a_0^\varepsilon(t)).$$

Owing to the positiveness of $\beta^\varepsilon(\cdot)$, $\delta g_\alpha(t) \neq 0$ only when the argument of $g_\alpha(\cdot)$ is less than some $C\alpha$ in absolute value. The properties of $y(\cdot)$ and $y_0(\cdot)$ imply that there is a real C such that $E|\delta g_\alpha(t)| \le C\alpha\varepsilon$. Partly due to this, the deletion of the $\delta g_\alpha(\cdot)$ in (A2) does not affect the limit, and we drop the term from (A2).

We now have the system $\dot{\hat\theta}^\varepsilon = Lg_\alpha(\beta^\varepsilon(t)a_0^\varepsilon(t))/\varepsilon$. For the final alteration, note first that the noise term in the argument of $g_\alpha(\cdot)$ above is $\frac{1}{\varepsilon}[\sigma_0^2+\sigma^2\cos(\omega_0 t+\hat\theta^\varepsilon(t))]^{-1/2}[\cos(\omega_0 t+\hat\theta^\varepsilon(t))y^\varepsilon(t)+y_0^\varepsilon(t)]$. Change time scale $t/\varepsilon^2 \to t$. As will be seen, the calculations which concern the shape of the limit are in the new time scale. In this scale, the noise term is

$$\frac{1}{\varepsilon}[\sigma_0^2+\sigma^2\cos(\omega_0\varepsilon^2 t+\hat\theta^\varepsilon(\varepsilon^2 t))]^{-1/2}[\cos(\omega_0\varepsilon^2\, t+\hat\theta^\varepsilon(\varepsilon^2 t))y(t)+y_0(t)].$$

If $\hat{\theta}^\varepsilon(\cdot)$ does not become "too wild - too fast" as $\varepsilon \to 0$, then for small ε the term has essentially the properties of a stationary Gaussian process $\tilde{y}(\cdot)$ with mean 0 and covariance e^{-at}, and we replace it by such a process. Such a facile argument does not justify the replacement (for the PLL case).

Dropping the $\omega_0\varepsilon^2 t$ part is unimportant and does not affect the result. Dropping the $\hat{\theta}^\varepsilon$ dependence is important, because it eliminates the correction term. Even without this alteration, $\{\hat{\theta}^\varepsilon(\cdot)\}$ can be shown to be tight* in $D[0,\infty)$, but the details are quite lengthy and will not be given. The main problem (apart from the length of the calculation) is that we have not been able to get a nice expression for the correction term. However, some formal and rough calculations suggest that its deletion does not substantially alter the system's main qualitative properties, and that the effects of the deleted term are small for large ω_0.

In the control theory problem of Figure 3 where the multiplier is replaced by an adder, the input noise is not multiplied by a state-dependent function at the input, the actual value of the input to the nonlinearity can be used in lieu of (A1) and there is no need to drop any terms. Our method is then rigorous.

Finally, the system to be analyzed is

$$\text{(A3)} \qquad \dot{\hat{\theta}}^\varepsilon(t) = \frac{L}{\varepsilon} g_\alpha(s(t,\hat{\theta}^\varepsilon(t)) + \tilde{y}^\varepsilon(t)/\varepsilon),$$

The terminology is that of weak convergence theory (Billingsley [7]). $D[0,\infty)$ is the space of real-valued functions which have left-hand limits and are right continuous, and which has an appropriate topology. By tightness, we mean that the measures that $\{\hat{\theta}^\varepsilon(\cdot)\}$ induces on $D[0,\infty)$ are weak sequentially compact. Later we use $C[0,T]$ and $C[0,\infty)$, the spaces of real-valued continuous functions (on $[0,T]$ or $[0,\infty)$) with the sup norm topology.

where $\tilde{y}^{\varepsilon}(t) = \tilde{y}(t/\varepsilon^2)$ and $s(t,\hat{\theta}^{\varepsilon}(t)) = A\beta^{\varepsilon}(t)\sin(\omega_0 t+\theta(t))$ $\cos(\omega_0 t+\hat{\theta}^{\varepsilon}(t))$.

Appendix 2

Derivation of the limit (as $\varepsilon \to 0$, $\alpha \to 0$) of the processes $\{\hat{\theta}^{\varepsilon}(\cdot)\}$ of (A3). Theorem 1 of [1] will be applied and most of the details of the following proposition will be given: $\{\hat{\theta}^{\varepsilon}(\cdot),\varepsilon\to 0,\alpha\to 0\}$ is tight and as $\alpha \to 0$ and $\varepsilon \to 0$, $\hat{\theta}^{\varepsilon}(\cdot)$ converges weakly† to the process $\hat{\theta}(\cdot)$ satisfying the diffusion equation (we set L = 1 for simplicity; the limit is linear in L)

$$\text{(A4)} \qquad d\hat{\theta}(t) = \sqrt{\frac{2}{\pi}}\,\frac{s(t,\hat{\theta}(t))}{\sigma}dt + \sqrt{\frac{2\ln 2}{a}}\,dB,$$

where

$$s(t,\hat{\theta}) = [\sigma_0^2+\sigma^2\cos(\omega_0 t+\hat{\theta})]^{-1/2}[A\sin(\omega_0 t+\theta(t))\cos(\omega_0 t+\hat{\theta})].$$

Then, in Appendix 3, it will be shown that as $\sigma_0 \to 0$, the solution of (A4) converges weakly to that of (3.3) and as $\omega_0 \to \infty$, the solution of (3.3) converges weakly to that of (3.5).

† In $D[0,\infty)$. Even though the paths are all continuous, it is convenient and traditional in the proofs of the basic theorems to assume that they are elements of a space of discontinuous functions $D[0,\infty)$. The reason is that the weak* sequential compactness of the associated measures is easier to prove there. This is purely for mathematical convenience and need not trouble us. Of course, the measures are concentrated on any measurable set in $D[0,\infty)$ which contains the continuous functions.

<u>The method</u>. The technique is close to that of [1, Section 6], the main difference being due to the fact that our $g_\alpha(\cdot)$ is not a limiter and that $s(t,\hat{\theta}^\varepsilon)$ has $\hat{\theta}^\varepsilon$ dependence. We briefly recall the definitions (from [1]) of A^ε and p-lim. Let E_t^ε denote expectation conditioned on $\{\tilde{y}^\varepsilon(u), u\le t\} = \{\tilde{y}(u), u\le t/\varepsilon^2\}$. Let $f^\varepsilon(\cdot)$ be (progressively measurable) functions of $\tilde{y}^\varepsilon(\cdot)$ such that $\sup_{t,\varepsilon} E|f^\varepsilon(t)| < \infty$, and $E|f^\varepsilon(t)| \to 0$ as $\varepsilon \to 0$, each t. Then we say that $\text{p-lim}_{\varepsilon\to 0} f = 0$. We also say $\text{p-lim}_{\delta\to 0} f^\delta(\cdot) = 0$ if for each $\delta > 0$, $f^\delta(\cdot)$ is a (progressively measurable) function of $\tilde{y}^\varepsilon(\cdot)$ and $\hat{\theta}^\varepsilon(\cdot)$, and $\sup_{t,\delta} E|f^\delta(t)| < \infty$ and $E|f^\delta(t)| \to 0$ as $\delta \to 0$. We say that $f^\varepsilon \in \mathscr{D}(\hat{A}^\varepsilon)$ and $\hat{A}^\varepsilon f^\varepsilon = q^\varepsilon$ if

$$\underset{\delta\to 0}{\text{p-lim}} \left[\frac{E_t^\varepsilon f^\varepsilon(t+\delta) - f^\varepsilon(t)}{\delta} - q^\varepsilon(t)\right] = 0$$

and $q^\varepsilon(\cdot)$ is p-right continuous. See [1] for more detail.

Let $\hat{\mathscr{C}}_0^{1,3}$ denote the space of real-valued functions on $R \times [0,\infty)$ with compact support and whose first three x-derivatives and first t-derivative are continuous. By Theorem 1 of [1], in order to prove the proposition, we need to prove tightness of $\{\hat{\theta}^\varepsilon(\cdot), \varepsilon\to 0, \alpha\to 0\}$ and for each $f(\cdot) \in \hat{\mathscr{C}}_0^{1,3}$, we need find a sequence $f^\varepsilon(\cdot) \in \mathscr{D}(\hat{A}^\varepsilon)$ such that

(A5)

$$\underset{\substack{\varepsilon\to 0\\ \alpha\to 0}}{\text{p-lim}} \left[f^\varepsilon(\cdot) - f(\hat{\theta}^\varepsilon(\cdot),\cdot)\right] = 0$$

$$\underset{\substack{\varepsilon\to 0\\ \alpha\to 0}}{\text{p-lim}} \left[\hat{A}^\varepsilon f^\varepsilon(\cdot) - \left(\frac{\partial}{\partial t} + A\right) f(\hat{\theta}^\varepsilon(\cdot),\cdot)\right] = 0,$$

where A is the infinitesimal operator of the diffusion (A4). As in [1], we will get $f^{\varepsilon}(\cdot)$ in the form $f^{\varepsilon}(\cdot) = f(\hat{\theta}^{\varepsilon}(\cdot),\cdot) + f_1^{\varepsilon}(t) + f_2^{\varepsilon}(t)$, where the f_i^{ε} will be defined below. First the $f^{\varepsilon}(\cdot)$ will be found, then tightness discussed. Even without tightness (A5) implies that the finite-dimensional distributions of (A3) converge to those of (A4) [1, Theorem 1].

Step 1. By the stationarity and distributional properties of $\tilde{y}(\cdot)$ and uniformity in s in bounded sets, we have

$$\frac{1}{\varepsilon} E\, g_{\alpha}(s+\tilde{y}^{\varepsilon}(t)/\varepsilon) = \frac{1}{\varepsilon}[P\{\tilde{y}(0) > -\varepsilon s+\varepsilon\alpha\} - P\{\tilde{y}(0) < -\varepsilon s-\varepsilon\alpha\}] + O(\varepsilon\alpha)/\varepsilon$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{s}{\sigma} + O(\alpha) + O(\varepsilon).$$

From this we can see the necessity of the $L_{\varepsilon} = L/\varepsilon$ scaling.

Step 2. Fix $f(\cdot,\cdot) \in \hat{\mathscr{C}}_0^{1,3}$. Then (subscripts x, t on $f(\cdot,\cdot)$ denote the derivatives with respect to the first and second arguments, resp.)

(A6) $\hat{A}^{\varepsilon} f(\hat{\theta}^{\varepsilon}(t),t) = f_t(\hat{\theta}^{\varepsilon}(t),t) + f_x(\hat{\theta}^{\varepsilon}(t),t) g_{\alpha}(s(t,\hat{\theta}^{\varepsilon}(t))+\tilde{y}^{\varepsilon}(t)/\varepsilon)/\varepsilon$

Continuing to follow the method of [1], $f_1^{\varepsilon}(\cdot)$ is to be introduced in a way that allows "cancellation or averaging out" of the $1/\varepsilon$ term in (A6).

Define the centered function (the expectation is over $\tilde{y}(0)$ only, $\hat{\theta}^{\varepsilon}$ is considered to be a parameter here) $\bar{g}_{\alpha}(\cdot)$ by

$$\bar{g}_{\alpha}(\hat{\theta}^{\varepsilon},t,\tilde{y}^{\varepsilon}(t)/\varepsilon) = g_{\alpha}(s(t,\hat{\theta}^{\varepsilon})+\tilde{y}^{\varepsilon}(t)/\varepsilon) - Eg_{\alpha}(s(t,\hat{\theta}^{\varepsilon})+\tilde{y}(0)/\varepsilon).$$

Now, define $f_1^\varepsilon(\cdot)$ by $f_1^\varepsilon(t) = f_1^\varepsilon(\hat\theta^\varepsilon(t),t)$ where

$$f_1^\varepsilon(\hat\theta,t) = \frac{1}{\varepsilon}\int_0^\infty E_t^\varepsilon f_x(\hat\theta,t+\tau)\bar g_\alpha(\hat\theta,t+\tau,\tilde y^\varepsilon(t+\tau)/\varepsilon)d\tau$$

$$= \varepsilon\int_0^\infty E_t^\varepsilon f_x(\hat\theta,t+\varepsilon^2\tau)\bar g_\alpha(\hat\theta,t+\varepsilon^2\tau,\tilde y(\tfrac{t}{\varepsilon^2}+\tau)/\varepsilon)d\tau.$$

Note that $\text{p-lim}_{\varepsilon\to 0}\, f_1^\varepsilon(\cdot) = 0$, owing to the centering about the mean, and the exponential correlation of the Gaussian process $\tilde y(\cdot)$. In fact, it will be shown that it follows from the estimate (A7).

(A7) on the set $\{|\tilde y(0)|\le 1\}$ or even on $\{|\tilde y(0)|\le e^{a\tau/2}\}$, $|P\{\tilde y(\tau)\in B|\tilde y(0)\} - P\{\tilde y(t)\in B\}| \le Ce^{-a_1\tau}$ for some $a_1 > 0$, uniformly in B. The 0 and τ arguments of $\tilde y(\cdot)$ can be replaced by t and $\tau+t$, resp. Similarly, for $\tau_2 > \tau_1 > 0$, $|P\{\tilde y(\tau_i)\in B_i,\ i=1,2|\tilde y(0)\} - P\{y_i(\tau_i)\in B_i,\ i=1,2\}| \le Ce^{-a_1\tau_i}$ for some $a_1 > 0$.

In the sequel the values of the constants $a_1 > 0$ and C may change from usage to usage.

To show (A7), we note that $(\tilde y(t),\tilde y(0))$ are Gaussian with mean zero and correlation exp -at, and compare the two distributions, the first of $\tilde y(\tau)$ (or $\tilde y(\tau_1),\tilde y(\tau_2)$) with initial condition $\tilde y(0)$, and the second the stationary distribution of $\tilde y(\tau)$ (or of $\tilde y(\tau_1),\tilde y(\tau_2)$).

Estimate (A7) implies that $|f_1^\varepsilon(t)| = O(\varepsilon)$ uniformly in ω on $\{|y^\varepsilon(t)|\le 1\}$ (the integrand of the second expression defining $f_1^\varepsilon(\cdot)$ goes to zero exponentially on that set by (A7)). Define $v_1 = \min\{\tau: e^{-a\tau/2}|\tilde{y}^\varepsilon(t)|\le 1\}$. Write

$$f_1^\varepsilon(\hat{\theta},t) = \varepsilon \int_0^{v_1} E_t^\varepsilon B d\tau + \varepsilon \int_{v_1}^\infty E_t^\varepsilon B d\tau,$$

where B is the argument of E_t^ε in the second expression above for $f_1^\varepsilon(\hat{\theta},t)$. The first term is bounded in absolute value by $\varepsilon C v_1$ and the integrand of the second by $C \exp -a_1(\tau - v_1)$, which is integrable. Thus

$$\text{(A8)} \qquad \begin{aligned} |f_1^\varepsilon(t)| &\le C\varepsilon[1+v_1] \\ &\le C\varepsilon[1 + \max(0, \log|\tilde{y}^\varepsilon(t)|)], \end{aligned}$$

which justifies the p-lim assertion above. To see how a mixing condition such as (A7) can be used to get bounds on functions such as $E_t^\varepsilon F(\tilde{y}(t+\tau))$ for bounded measurable $F(\cdot)$ satisfying $EF(\tilde{y}(t)) \equiv 0$, in more general cases, see [9, Lemma 1].

<u>Step 3</u>. It can be verified that (for some version, as always) $f_1^\varepsilon(\cdot) \in \mathscr{D}(\hat{A}^\varepsilon)$ and that

$$\text{(A9)}\quad \hat{A}^{\varepsilon} f_1^{\varepsilon}(t) = -\frac{1}{\varepsilon} f_x(\hat{\theta}^{\varepsilon}(t),t)\bar{g}_{\alpha}(\hat{\theta}^{\varepsilon}(t),t,\tilde{y}^{\varepsilon}(t)/\varepsilon)$$

$$+\frac{1}{\varepsilon}\int_0^{\infty} d\tau\ E_t^{\varepsilon}[f_x(\hat{\theta}^{\varepsilon}(t),t+\tau)\bar{g}_{\alpha}(\hat{\theta}^{\varepsilon}(t),t+\tau,\tilde{y}^{\varepsilon}(t+\tau)/\varepsilon)]_x$$

$$\cdot\ g_{\alpha}(s(t,\hat{\theta}^{\varepsilon}(t))+\tilde{y}^{\varepsilon}(t)/\varepsilon)/\varepsilon.$$

The first term of $\hat{A}^{\varepsilon} f_1^{\varepsilon}(t) = \hat{A}^{\varepsilon} f_1^{\varepsilon}(\hat{\theta}^{\varepsilon}(t),t)$ is (loosely) the component of the derivative with respect to t, when $\hat{\theta}^{\varepsilon}(\cdot)$ is held constant, and the second term contains the contributions to the derivative due to the $\hat{\theta}^{\varepsilon}(\cdot)$ variations only. The (non-mean) part of the first term on the right cancels the second term of (A6). This is why $f_1^{\varepsilon}(\cdot)$ was defined as it was.

The second term of (A9) has two components which, after a change of variable $\tau/\varepsilon^2 \to \tau$, can be written as ($\bar{g}_{\alpha,x}$ denotes the derivative of $\bar{g}_{\alpha}(\cdot)$)

$$\text{(A10)}\quad \int_0^{\infty} d\tau\ E_t^{\varepsilon} f_{xx}(\hat{\theta}^{\varepsilon}(t),t+\varepsilon^2\tau)\bar{g}_{\alpha}(\hat{\theta}^{\varepsilon}(t),t+\varepsilon^2\tau,\tilde{y}(\frac{t}{\varepsilon^2}+\tau)/\varepsilon)$$

$$\cdot\ g_{\alpha}(s(t,\hat{\theta}^{\varepsilon}(t))+\tilde{y}(t/\varepsilon^2)/\varepsilon)\ +$$

$$+\int_0^{\infty} d\tau\ E_t^{\varepsilon} f_x(\hat{\theta}^{\varepsilon}(t),t+\varepsilon^2\tau)\bar{g}_{\alpha,x}(\hat{\theta}^{\varepsilon}(t),t+\varepsilon^2\tau,\tilde{y}(\frac{t}{\varepsilon^2}+\tau)/\varepsilon)$$

$$\cdot\ g_{\alpha}(s(t,\hat{\theta}^{\varepsilon}(t))+\tilde{y}(t/\varepsilon^2)/\varepsilon).$$

Step 4. Both terms in (A10) exist by the same arguments which lead to (A8). We wish to show that the second term of (A10) is negligible, as well as get an estimate which is useful for the tightness argument. The fact that the support of $g_{\alpha,x}(\cdot)$ is in $[-\alpha,\alpha]$ and that $|g_{\alpha,x}(x)| \leq C/\alpha$ and the uniform boundedness of $s(t,\hat{\theta}^{\varepsilon}(t))$ will be used heavily. Let $I\{A\}$ denote the indicator function of the set A.

By the use of (A7), it can be verified that (the bounds below are uniform in s in bounded sets)

$$\text{(A11)} \quad |E_t^{\varepsilon}\bar{g}_{\alpha,x}(s+\tilde{y}(\frac{t}{\varepsilon^2}+\tau)/\varepsilon)| \leq [\exp -a_1\tau + I\{|\tilde{y}^{\varepsilon}(t)|>e^{a\tau/2}\}]C/\alpha.$$

We need a bound which goes to zero as $\varepsilon \to 0$. Proceeding to get this, we have

$$\text{(A12)} \quad P\{|s+\tilde{y}(\frac{t}{\varepsilon^2}+\tau)/\varepsilon| \leq \alpha | \tilde{y}^{\varepsilon}(t)=y_0\} = O(\alpha\varepsilon)$$

uniformly in $|y_0| \geq 1$ and $\tau \geq 0$.

The estimate (A12) and the fact that the support of $g_{\alpha,x}(\cdot)$ is in $[-\alpha,\alpha]$ and that it's bounded by C/α yield that the left side of (A11) is also bounded above by $O(\varepsilon)$, uniformly in $|\tilde{y}^{\varepsilon}(t)| \geq 1$, $\tau \geq 0$. This† and (A11) yield that on the set $|\tilde{y}^{\varepsilon}(t)| \geq 1$ the left side of (A11) is bounded above by

$$\text{(A13)} \quad [\exp -a_1\tau + I\{|\tilde{y}^{\varepsilon}(t)|>e^{a\tau/2}\}]^{1/2}\, C(\varepsilon/\alpha)^{1/2}.$$

†Note that $|x| \leq a$, $|x| \leq b$ imply $|x| \leq \sqrt{ab}$.

Integrating (A13) with respect to τ in $[0,\infty)$ and noting the boundedness of g_α and f_x yields the upper bound

$$C[1 + \max(0, \log|\tilde{y}^\varepsilon(t)|)]\ (\varepsilon/\alpha)^{1/2}$$

on the second term of (A10) when $|\tilde{y}^\varepsilon(t)| \geq 1$.

In order to get a suitable bound for the case $|\tilde{y}^\varepsilon(t)| \leq 1$, split the integral of the second term in (A10) as

$$\int_0^\varepsilon + \int_\varepsilon^\infty .$$

The $\int_0^\varepsilon$ is clearly $O(\varepsilon/\alpha)$. For the other term note that for $\tau \geq \varepsilon$, the density of $\tilde{y}(\frac{t}{\varepsilon^2} + \varepsilon)$, conditioned on any value of $|\tilde{y}(t/\varepsilon^2)|$ in $[0,1]$, is $\leq O(1/\sqrt{\varepsilon})$. Thus (A12) holds in this case ($\tau \geq \varepsilon$, $|\tilde{y}(t/\varepsilon^2)| \leq 1$), but with $O(\alpha\varepsilon)$ replaced by $O(\alpha\sqrt{\varepsilon})$. Again, combining these estimates with (A11) yields that the left side of (A11) is bounded above by (A13) also when $(|\tilde{y}^\varepsilon(t/\varepsilon^2)| \leq 1)$, but with $(\varepsilon/\alpha)^{1/2}$ replaced by $\varepsilon^{1/4}/\alpha^{1/2}$. Integrating the bound with respect to τ in $[0,\infty)$ yields the upper bound to the second term of (A10):

(A14) $\qquad C[1 + \max(0, \log|\tilde{y}^\varepsilon(t)|)]\,\varepsilon^{1/4}/\alpha^{1/2}.$

Clearly $\text{p-lim}_{\varepsilon\to0,\alpha\to0}$(second term of (A10)) = 0.

Step 5. Turning our attention to the first term of (A10), it will be "averaged out" by use of a $f_2^\varepsilon(\cdot)$. This "averaging out" will give use the second-order term of the operator of the limit process. Define $A_0^\varepsilon f(\hat{\theta},t)$ to be the expectation of the first term of (A10) when $\hat{\theta}^\varepsilon(t)$ is replaced by the parameter θ. Using the stationarity of $\tilde{y}(\cdot)$, we have

$$A_0^\varepsilon f(\hat{\theta},t) = \int_0^\infty E\, f_{xx}(\hat{\theta},t+\varepsilon^2\tau)\bar{g}_\alpha(\hat{\theta},t+\varepsilon^2\tau,\tilde{y}(\tau)/\varepsilon)g_\alpha(s(t,\hat{\theta})+\tilde{y}(0)/\varepsilon)d\tau$$

$$= \frac{1}{\varepsilon^2}\int_0^\infty E\, f_{xx}(\hat{\theta},t+\tau)\bar{g}_\alpha(\hat{\theta},t+\tau,\frac{\tilde{y}^\varepsilon(t+\tau)}{\varepsilon})g_\alpha(s(t,\hat{\theta})+\tilde{y}(\frac{t}{\varepsilon^2})/\varepsilon)d\tau.$$

Define $f_2^\varepsilon(t) = f_2^\varepsilon(\hat{\theta}^\varepsilon(t),t)$ by

$$\text{(A15)}\quad f_2^\varepsilon(\hat{\theta},t) = \int_0^\infty dv\{\int_0^\infty d\tau[f_{xx}(\hat{\theta},t+\tau+v)E_t^\varepsilon\, \frac{\bar{g}_\alpha}{\varepsilon}(\hat{\theta},t+\tau+v\ ,\tilde{y}(\frac{t+\tau+v}{\varepsilon^2})/\varepsilon)\,\cdot$$

$$\frac{g_\alpha}{\varepsilon}(s(t+v,\hat{\theta})+\tilde{y}(\frac{t+v}{\varepsilon^2})/\varepsilon)] - A_0^\varepsilon f(\hat{\theta},t+v)\} \equiv \int_0^\infty dv\, F_\varepsilon(t+v).$$

Note that when $v = 0$ the inner $\{\}$ term $\{F_\varepsilon(t)\}$ is just the first term of (A10) (with a change of variable) minus its exponentiation and with $\hat{\theta}^\varepsilon(t)$ replaced by $\hat{\theta}$. By a change of variables $\tau/\varepsilon^2 \to \tau$, $v/\varepsilon^2 \to v$, and an application of (A7) similar to that used to get the bound on $|f_1^\varepsilon(t)|$, we get that the inner integral $F_\varepsilon(t+v)$ exists for each v. We will soon show that the double integral exists. Define

$$H_\varepsilon(\hat{\theta},t,\tau,v) = f_{xx}(\hat{\theta},t+\varepsilon^2\tau+\varepsilon^2 v)$$

$$= \bar{g}_\alpha(\hat{\theta},t+\varepsilon^2\tau+\varepsilon^2 v,\tilde{y}(\frac{t}{\varepsilon^2}+\tau+v)/\varepsilon)g_\alpha(s(t+\varepsilon^2 v,\hat{\theta}),\tilde{y}(\frac{t}{\varepsilon^2}+v)/\varepsilon).$$

An alternative representation (using the above-mentioned change of variables) of $f_2^\varepsilon(\hat{\theta},t)$ is

$$\text{(A16)} \qquad \varepsilon^2 \int_0^\infty dv \int_0^\infty d\tau [E_t^\varepsilon H_\varepsilon(\hat{\theta},t,\tau,v) - EH_\varepsilon(\hat{\theta},t,\tau,v)].$$

Denote the integrand by $E_t^\varepsilon B$. The inner integral still exists, of course - we only changed variables. Recall $v_1 = \min\{v: e^{-av/2}|\tilde{y}(t/\varepsilon^2)|\leq 1\}$. Write the last integral as

$$\varepsilon^2 \int_{v_1}^\infty dv \int_0^\infty d\tau\, E_t^\varepsilon B + \varepsilon^2 \int_0^{v_1} dv \int_0^\infty d\tau\, E_t^\varepsilon B = II + I.$$

Let us now evaluate II. By (A7) and the definition of v_1, the absolute value of the integral in II is bounded above by some $C \exp -a_1(v-v_1)$, $a_1 > 0$. Also, $|E\, H_\varepsilon(\hat{\theta},t,\tau,v)| \leq C \exp -a_1\tau$ for some $a_1 > 0$. Furthermore, by (A7) and $v \geq v_1$, and for some $C > 0$, $a_1 > 0$ (whose values may again change from usage to usage)

$$|E_t^\varepsilon\, H_\varepsilon(\hat{\theta},t,\tau,v)| \leq E_t^\varepsilon|E^\varepsilon_{t+\varepsilon^2 v} H_\varepsilon(\hat{\theta},t,\tau,v)|$$

$$\leq C\, E_t^\varepsilon[\exp -a_1\tau + I\{e^{-a\tau/2}|\tilde{y}(\tfrac{t}{\varepsilon^2}+v)|\geq 1\}]$$

$$= C \exp -a_1\tau + C\, P\{|y(\tfrac{t}{\varepsilon^2}+v)|\geq e^{a\tau/2}|v\geq v_1,\tilde{y}(u),u\leq t/\varepsilon^2\}$$

$$\leq C \exp -a_1\tau + C\, e^{-a\tau/2}\, E\{|\tilde{y}(\tfrac{t}{\varepsilon^2}+v)||v\geq v_1,\tilde{y}(u),u\leq t/\varepsilon^2\}$$

$$\leq C \exp -a_1\tau.$$

By combining the above estimates, we get that the integral in II is bounded in absolute value by C exp $-a_1(v-v_1+\tau)$ and, consequently, II = $O(\varepsilon^2)$.

The component I is also $O(\varepsilon^2)$ but not uniformly in $\tilde{y}^\varepsilon(t)$. Bound the inner integral of I by

$$\left|\int_0^\infty d\tau\, E_t^\varepsilon B\right| \leq E_t^\varepsilon \int_0^\infty d\tau \left|E^\varepsilon_{t+\varepsilon^2 v} B\right| \equiv \text{III}.$$

By the arguments connected with the bound on $|f_1^\varepsilon(t)|$, we get the bound

$$\text{III} \leq C\varepsilon^2 E_t^\varepsilon[1 + \max(0, \log|\tilde{y}(\tfrac{t}{\varepsilon^2}+v)|)].$$

Using the concavity of max(0, log|y|),

$$\text{III} \leq C\varepsilon^2[1 + \max(0, \log|\tilde{y}^\varepsilon(t)|)].$$

Since $v_1 \leq C \max(0, \log|\tilde{y}^\varepsilon(t)|)$, we have that

(A17) $$|f_2^\varepsilon(t)| \leq C\varepsilon^2[1 + \max(0,\log|\tilde{y}^\varepsilon(t)|]^2.$$

Furthermore, it can be shown that $f_2^\varepsilon(\cdot) \in \mathscr{D}(\hat{A}^\varepsilon)$ and that $\hat{A}^\varepsilon f_2^\varepsilon(\hat{\theta}^\varepsilon(t),t) \equiv \hat{A}^\varepsilon f_2^\varepsilon(t)$ equals the term obtained by holding $\hat{\theta}^\varepsilon(\cdot)$ constant plus the term obtained by holding t constant (in the calculation of the "derivative" of $f_2^\varepsilon(\hat{\theta}^\varepsilon(\cdot),\cdot)$. In particular,

$\hat{A}^{\varepsilon} f_2^{\varepsilon}(t) = [\text{negative of first term of (A10)} + A_0^{\varepsilon} f(\hat{\theta}^{\varepsilon}(t),t)]$

$$+ (\text{term whose } \underset{\substack{\varepsilon\to 0\\ \alpha\to 0}}{\text{p-lim}} \text{ equal zero}).$$

The first term is, of course, just $-F_{\varepsilon}(t)$. The latter (p-lim = 0) term is $f_{2,x}^{\varepsilon}(\hat{\theta},t) g_{\alpha}(s(t,\hat{\theta}),t,\tilde{y}^{\varepsilon}(t)/\varepsilon)/\varepsilon = f_{2,x}^{\varepsilon}(\hat{\theta},t)\dot{\hat{\theta}}$, where $\hat{\theta} = \hat{\theta}^{\varepsilon}(t)$ and the $f_{2,x}^{\varepsilon}$ is the derivative with respect to the $\hat{\theta}$ argument. It can be shown that this derivative can be taken under the integral sign in (A15). The components of the derivative which involve f_{xxx} are uniformly bounded by $O(\varepsilon)$. The other components are of the form

$$\varepsilon\iint [E_t^{\varepsilon} f_{xx}\bar{g}_{\alpha,x} g_{\alpha} + E_t^{\varepsilon} f_{xx}\bar{g}_{\alpha} g_{\alpha,x} - E f_{xx}\bar{g}_{\alpha,x} g_{\alpha} - E f_{xx}\bar{g}_{\alpha} g_{\alpha,x}]d\tau dv\ g_{\alpha}$$

with the obvious arguments for the functions. It is treated similarly to the way (A16) was treated and is bounded by (A17) but with ε/α replacing ε^2.

<u>Step 6</u>. By combining the above estimates with $f^{\varepsilon} = f + f_1^{\varepsilon} + f_2^{\varepsilon}$ ($\partial/\partial x$ denotes $\partial/\partial\hat{\theta}$, as usual),

$$\text{p-lim}[f^{\varepsilon}(t) - f(\hat{\theta}^{\varepsilon}(t),t)] = 0$$

$$\text{p-lim}[\hat{A}^{\varepsilon} f^{\varepsilon}(t) - (\frac{\partial}{\partial t} + \sqrt{\frac{2}{\pi}}\ \frac{s(t,\hat{\theta}^{\varepsilon}(t))}{\sigma}\ \frac{\partial}{\partial x} + A_0^{\varepsilon}) f(\hat{\theta}^{\varepsilon}(t),t)] = 0.$$

A very similar proof to that of [1, Section 6, part 2] yields that $A_0^{\varepsilon} f(\hat{\theta},t) \to f_{xx}(\hat{\theta},t) \ln 2/a$ uniformly in $\hat{\theta}$ for each t. Then, by [1, Theorem 1] (see the part of Appendix 2 above step 1), the finite-dimensional distributions of $\{\hat{\theta}^{\varepsilon}(\cdot)\}$ converge to those of the Markov process $\hat{\theta}(\cdot)$ with infinitesimal operator

$$\sqrt{\frac{2}{\pi}}\,\frac{s(t,\hat{\theta}(t))}{\sigma}\,\frac{\partial}{\partial x}+\frac{\ln 2}{a}\,\frac{\partial^2}{\partial x^2}\;;$$

i.e., to (A4). This convergence of finite-dimensional distributions is good enough for many applications.

Step 7. Tightness. We skimp on details. Either [11, Lemma 1] or [10, Theorem 2] can be used. The boundedness required in [10, Theorem 2], can be dropped here, since finite-dimensional distributions converge (for justification, see the remarks on p. 628 of [12]). The estimates that we need for the use of either of these theorems are supplied by (A14), (A17): namely, we need that for each $T > 0$

$$\lim_{N\to\infty}\ \varlimsup_{\substack{\varepsilon\to 0\\ \alpha\to 0}}\ P\{\sup_{t\le T}|\hat{A}^{\varepsilon}f^{\varepsilon}(t)| \ge N\} = 0,$$

$$\lim_{\varepsilon\to 0} P\ \sup_{t\le T}\{f_1^{\varepsilon}(t)+f_2^{\varepsilon}(t)| \ge \delta > 0\} = 0, \qquad \text{each } \delta > 0,$$

all of which can be proved from (A14), (A17), since for any constant $\gamma > 0$, the Gaussianness and stationarity imply that $\lim_{\varepsilon\to 0}\sup_{t\le T}\varepsilon^{\gamma}|\tilde{y}(t/\varepsilon^2)| = 0$ w. p. 1.

Appendix 3

Getting from (A4) to (3.3) and then to (3.5) is much easier than getting (A4), since we are working with Itô equations. Over any finite interval [0,T], the measures in C[0,T] induced by (A3) are absolutely continuous with respect to Wiener measure, uniformly in σ_0 and ω_0 [8]. Thus the sequence given by (A4) and parametrized by σ_0^2 is tight in $C[0,\infty)$. Since the measure of any limit process must also be absolutely continuous with respect to Wiener measure, it can be shown that any limit must have the form (3.3) even though the drift is discontinuous. A "Skorokhod imbedding" technique can be used to fill in the details; i.e., we can choose a weakly convergent subsequence with limit denoted by $\hat{\theta}(\cdot)$ and we choose the probability space so that $\hat{\theta}(\cdot;\sigma_0) \to \hat{\theta}(\cdot)$ uniformly as $\sigma_0 \to 0$ on each interval [0,T], w. p. 1. The form of the limit of the chosen convergent subsequence can easily be seen to be (3.3) by this operation. There can only be one limit to the original sequence, since all limits have the form (3.3) and the solution to (3.3) is unique in the sense of measure.

Now, we go from (3.3) to (3.5) as $\omega_0 \to \infty$. Again, Theorem 1 of [1] is used. As noted in the previous paragraph, the solutions of (3.3) (parametrized by ω_0) are tight in $C[0,\infty)$. Write $\omega_0 = p$ and denote the corresponding solution to (3.3) by $\hat{\theta}^p$. Then, for each $f(\cdot,\cdot) \in \hat{\mathscr{C}}_0^{1,3}$, we need only find a sequence $f^p(\cdot) \in \mathscr{D}(\hat{A}^p)$ for which (A5) (ε, α replaced by p) holds. Define (subscripts t, x denote the appropriate derivatives with respect to the second and first argument of $f(\cdot,\cdot)$, resp.)

$$f^p(t) = \frac{p}{2\pi}\int_0^{2\pi/p} E_t^p\, f(\hat{\theta}^p(t+s),t)\,ds,$$

where E_t^p denotes conditioning on $\hat{\theta}^p(u)$, $u \leq t$. Then $f^p(\cdot) \in \mathscr{D}(\hat{A}^p)$ and

$$\begin{aligned}\hat{A}^p f^p(t) &= \frac{p}{2\pi}\int_0^{2\pi/p} E_t^p f_t(\hat{\theta}^p(t+s),t)\,ds \\ &\quad + \frac{E_t^p f(\hat{\theta}^p(t+2\pi/p),t) - f(\hat{\theta}^p(t),t)}{2\pi/p} \\ &= S_1^p(t) + S_2^p(t).\end{aligned}$$

Because of the boundedness of the drift coefficient in (3.3) and the absolute continuity with respect to Wiener measure,

(A18) $E|\hat{\theta}^p(t+s)-\hat{\theta}^p(t)| \to 0$ as $s \to t$, uniformly in t, p.

Thus

$$\underset{p\to\infty}{\text{p-lim}}\,[f^p(t)-f(\hat{\theta}^p(t),t)] = 0,$$

$$\underset{p\to\infty}{\text{p-lim}}\,[S_1^p(t)-f_t(\hat{\theta}^p(t),t)] = 0.$$

We need only evaluate the limit of $S_2^p(\cdot)$. By applying Itô's lemma to $S_2^p(\cdot)$ and the process (3.3),

$$S_2^p(t) = \frac{p}{2\pi}\int_0^{2\pi/p} E_t^p f_{xx}(\hat{\theta}^p(t+s),t)\ \frac{\ln 2}{a}\ ds + \frac{p}{2\pi}\int_0^{2\pi/p} E_t^p f_x(\hat{\theta}^p(t+s),t)\ \cdot$$

$$\cdot\ \left[\sqrt{\frac{2}{\pi}}\ \frac{\sin(\omega_0(t+s)+\hat{\theta}^p(t+s))\,\mathrm{sign}[\cos(\omega_0(t+s)+\hat{\theta}^p(t+s))]}{\sigma}\right] ds$$

$$= T_1^p(t) + T_2^p(t).$$

Also, $\text{p-lim}_{p\to\infty}\ [T_1^p(t)-f_{xx}(\hat{\theta}^p(t),t)\ \frac{\ln 2}{a}] = 0.$

We need only check that $T_2^p(\cdot)$ has the correct ($\text{p-lim}_{p\to\infty}$ sense) limit. Using the Fourier series introduced in Section 3.1, write $T_2^p/(\sqrt{\frac{2}{\pi}}\ \frac{1}{\sigma})$ in the form

$$\frac{p}{2\pi}E_t^p \int_0^{2\pi/p} f_x(\hat{\theta}^p(t+s),t)\frac{2}{\pi}\ \sin(\theta(t+s)-\hat{\theta}^p(t+s))ds$$

$$+ \frac{p}{2\pi}\ E_t^p \int_0^{2\pi/p} ds\ \ f_x(\hat{\theta}^p(t+s),t)\frac{2}{\pi}\Big[\sin(2p(t+s)+\theta(t+s)+\hat{\theta}^p(t+s))$$

$$+ 2\ \sin(p(t+s)+\theta(t+s))\sum_{k=1}^{N}\ \frac{(-1)^k}{(2k+1)}\ \cos(2k+1)(p(t+s)+\hat{\theta}^p(t+s))\Big]$$

$$+ \frac{p}{2\pi}\ E_t^p \int_0^{2\pi/p} ds\ \ f_x(\hat{\theta}^p(t+s),t)\frac{4}{\pi}[\sin(p(t+s)+\theta(t+s))\ \cdot$$

$$\cdot\ \sum_{k=N+1}^{\infty}\frac{(-1)^k}{(2k+1)}\ \cos(2k+1)(p(t+s)+\hat{\theta}^p(t+s))]$$

$$= I_1^p(t) + I_2^p(t) + I_3^p(t).$$

As earlier,

$$\underset{p\to\infty}{\text{p-lim}}\ [I_1^p(t)-f_x(\hat{\theta}^p(t),t)\ \frac{2}{\pi}\ \sin(\theta(t)-\hat{\theta}^p(t))] = 0.$$

By (A18), we can replace $\hat{\theta}^p(t+s)$ by $\hat{\theta}^p(t)$ in $I_2^p(t)$ without altering the p-lim. By doing this, we see that $\text{p-lim}_{p\to\infty}\ I_2^p(t) = 0$ for each N.

For each $T_0 > 0$, $T_0 < T < \infty$, there is a finite measure which (uniformly in p and $t \in [T_0,T]$) dominates the measure of $\hat{\theta}^p(t)$. Using this and the fact that (the mean square value of the tail of the Fourier series of $\text{sign}[\cos(p(t+s)+y] \equiv q(s)$ over $[t,t+p/2\pi]$)

$$\frac{p}{2\pi}\int_0^{2\pi/p} \Big|\sum_{k=N+1}^{\infty} \frac{(-1)^k}{(2k+1)} \cos(2k+1)(p(t+s)+y)\Big|^2 ds \to 0$$

uniformly in t, y and p, as $N \to \infty$, yields that $E|I_3^p(t)|$ can be made as small as desired by making N large. This and $\text{p-lim}\ I_2^p = 0$, each N, yields $\text{p-lim}[I_2^p + I_3^p] = 0$.

Next, choose a weakly convergent subsequence of the process $\{\hat{\theta}^p(t),\ t \geq T_0 > 0\}$. Then $\hat{\theta}^p(T_0)$ converges weakly to a random variable $\hat{\theta}(T_0)$. As $T_0 \to 0$, the uniform absolute continuity with respect to Wiener measure on bounded time intervals implies that $\hat{\theta}(T_0)$ must converge to $\hat{\theta}^p(0) = \hat{\theta}(0)$ weakly as $T_0 \to 0$. This and the uniqueness of the solution to (3.5) imply that $\{\hat{\theta}^p(\cdot)\}$ converges weakly to (3.5).

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